ECO 2901
EMPIRICAL INDUSTRIAL ORGANIZATION

Lecture 4:
Static games of incomplete information with non-equilibrium beliefs: Model and Identification

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January 30, 2020

Lecture 4: Games of incomplete information with non-equilibrium beliefs

[1] Introduction

[2] Quick review of recent empirical evidence on firms' biased beliefs

[3] Model

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1. Introduction

Firms' Information & Beliefs

- In oligopoly markets, a firm's behavior depends on its **beliefs about the behavior of other firms** in the market.
- Firms form their beliefs under **uncertainty and asymmetric information**.
- Firms are different in their **ability for collecting and processing information**, for similar reasons as they are heterogeneous in their costs of producing goods and services.
- We expect firms to be **heterogeneous in their beliefs**.
- This heterogeneity has implications on their performance and on market outcomes.

Possible sources of firms' biased beliefs

- In reality, firms can face substantial uncertainty about other competitors' strategies.

- There are different sources of bias in players beliefs:

(a) **Limited information / attention**: *Some players do not have information about variables that are know to other players.*

(b) **Bounded rationality**: *Limited capacity to process information / compute;*

(c) **Strategic uncertainty:** *With multiple equilibria, players can have different beliefs about the selected equilibrium. Some players believe that they are playing equilibrium A, other players believe they are playing equilibrium B, . . .*

Relaxing Firms' Rational Beliefs

- Despite these arguments, in most fields in economics, and in particular in empirical IO, the status quo has been to **assume rational expectations**.
- There are good reasons to impose assumption of equilibrium beliefs:

  (a) This assumption has identification power.
  (b) Counterfactual analysis: model predicts how beliefs change endogenously.

- But it can be unrealistic in some applications, and can imply serious biases in our views on firms' competition.
- In these lectures, we will review some recent structural empirical papers of oligopoly competition that **relax the assumption of firms' rational beliefs**.

2. Quick Review of Recent Empirical Evidence on Firms' Biased Beliefs

US Telecommunication industry after deregulation

- **Goldfarb and Xiao (AER, 2011)** study entry decisions into local US telecommunication markets following the deregulatory Telecommunications Act of 1996, which allowed free competition.
- Holding other market characteristics constant, more experienced and better educated managers have a lower propensity to enter (and a lower propensity to exit after entry) into very competitive markets.
- This suggests that **better-educated managers are better at predicting competitors' behavior**.
- This hypothesis is confirmed from the estimation of a structural game of market entry with Cognitive Hierarchy beliefs.

Learning to bid after market deregulation

- **Doraszelski, Lewis, and Pakes (AER, 2018)** investigate firms' learning about competitors' bidding behavior just after the deregulation of the UK electricity market.
- In the first year after deregulation, **firms' bidding behavior was very heterogeneous and firms made frequent and sizable adjustments** in their bids.
- During the second year, there is a dramatic reduction in the range of bids. After three years, firms' bids become very stable.
- During these three phases, demand and costs were quite stable.
- The authors argue that the changes in firms' bidding strategies can be attributed to strategic uncertainty and learning..

Learning to price after market deregulation

- **Huang, Ellickson, and Lovett (2018)** study firms' price setting behavior in the Washington State liquor market following the privatization of the market in 2012.
- After liberalization, grocery chains newly entered the market. How did these new entrants learn about demand and learn to price optimally over time?
- The authors document **large and heterogeneous price movements in the first two years after the privatization**.
- The authors present evidence consistent with firms' learning about the idiosyncratic and common components of the demand shocks, and about the time persistence of these shocks.

Entry in the early years of UK fast-food restaurant industry

- **Aguirregabiria and Magesan (REStud, 2019)** study competition in store location between McDonalds (MD) and Burger King (BK) during the early years of the fast-food restaurant industry in the UK.
- Reduced form evidence shows that the number of own stores has a strong negative effect on the probability that BK opens a new store but **the effect of the competitor's number of stores is economically negligible**.
- This behavior cannot be rationalized by an equilibrium model of market entry where firms have equilibrium beliefs about the behavior of competitors.

Bidding behavior in the Texas electricity spot market

- **Hortacsu and Puller (RAND, 2008)** analyze firms' bidding behavior in the Texas electricity spot market.
- Their dataset contains detailed information not only on firms' bids but also on their marginal costs. Using these data, the authors construct the equilibrium bids of the game and compare them to the actual observed bids.
- They find statistically and **economically very significant deviations between equilibrium and actual bids**.
  Small firms don't supply much power even when is profitable to do so.
- This finding is consistent with low strategic ability in the bidding departments of small firms.
- This suboptimal behavior leads to significant efficiency losses.

Key Question

- A key issue in all these applications is how to find convincing evidence that (some) firms have non-equilibrium or biased beliefs, and this is not just **an artifact from the specification (or misspecification) of the model**.
- How can we be (more or less) confident that what we call bias beliefs cannot be explained by observable or unobservable variables affecting firms' demand or costs?
- To answer these questions, we need to study formally the identification of beliefs and structural parameters in profits in our model.
- This is the focus of most of the remaining part of this lecture.

3. Model

Model: Some key features

- **General model of market competition with firm's incomplete information**, that includes:
  - discrete and continuous choice games;
  - static and dynamic games;
  - Bertrand, Cournot, Auctions, Entry models ...
- A key feature of the model is that firms' beliefs are **unrestricted nonparametric functions of firms' information**. Profit function is also nonparametric.
- We will study the **identification of firms' beliefs and structural functions** (demand and costs) under different type of data in IO and using standard exclusion restrictions.
- For the moment, we focus on **static games.** Later we will extend this framework to **dynamic games**.

Profit function

- N firms competing in a market. The profit function of firm i:

$$\Pi_i(a_i, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x})$$

a_i is is the action of firm i
$\mathbf{a}_{-i}$ is the vector with the actions of the other firms
$\mathbf{x}$ represents variables that are common knowledge
ε_i is private information of firm i

- Firms' types $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_N)$ are drawn from a distribution F.

- Firms choose simultaneously their actions a_i to maximize their respective expected profits.

Beliefs

- A firm does not know the private information of its competitors and therefore it does not know their actions.
- Firms form probabilistic beliefs about the actions of competitors.
- Let $B_i(\mathbf{a}_{-i} \mid \varepsilon_i, \mathbf{x})$ be a probability density function that represents the belief of firm i.

Best response

- Given its beliefs, a firm's expected profit is:

$$\Pi_i^e(a_i, \varepsilon_i, \mathbf{x}; B_i) \quad = \quad \int \pi_i(a_i, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x}) \; B_i(\mathbf{a}_{-i}|\varepsilon_i, \mathbf{x}) \; d\mathbf{a}_{-i}$$

- A firm chooses its strategy function $\sigma_i(\varepsilon_i, \mathbf{x}; B_i)$, to maximize expected profits:

$$\sigma_i(\varepsilon_i, \mathbf{x}; B_i) = \arg\max_{a_i \in \mathcal{A}} \; \Pi_i^e(a_i, \varepsilon_i, \mathbf{x}; B_i)$$

- We can represent a firm's strategy as a **cumulative choice probability function**.

$$P_i\left(a_i \mid \mathbf{x}\right) \equiv \int 1\left\{\sigma_i(\varepsilon_i, \mathbf{x}; B_i) \leq a_i\right\} \; dF_i(\varepsilon_i|\mathbf{x})$$

Characterization of best response strategies

- Let $\Delta\Pi_i(a_i, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x})$ be the **marginal profit function** (discrete or continuous)

  Discrete choice:
  $\Delta\Pi_i(a_i, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x}) \equiv \Pi_i(a_i, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x}) - \Pi_i(a_i - 1, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x})$.

  Continuous choice: $\Delta\Pi_i(a_i, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x}) \equiv \dfrac{\partial\Pi_i(a_i, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x})}{\partial a_i}$.

- **ASSUMPTION 1:** $\Delta\Pi_i(a_i, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x})$ is strictly monotonic in a_i and ε_i.

- A stronger version of monotonicity in ε_i :

$$\Delta\Pi_i(a_i, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x}) = \Delta\pi_i(a_i, \mathbf{a}_{-i}, \mathbf{x}) - \varepsilon_i$$

- **ASSUMPTION 2:** (A) Independent private values ε_i. (B) F_i strictly increasing in $\mathbb{R}$.

Characterization of best response strategies [2]

- Under Assumptions 1 and 2 (with additivity of ε_i), for **continuous choice game**:

$$\int \Delta\pi_i(a_i, \mathbf{a}_{-i}, \mathbf{x})\ B_i(\mathbf{a}_{-i}|\mathbf{x})\ d\mathbf{a}_{-i} - \varepsilon_i = 0$$

- For **discrete choice game**:

$$\int \Delta\pi_i(a_i, \mathbf{a}_{-i}, \mathbf{x})\ B_i(\mathbf{a}_{-i}|\mathbf{x})\ d\mathbf{a}_{-i} - \varepsilon_i \geq 0$$

and

$$\int \Delta\pi_i(a_i + 1, \mathbf{a}_{-i}, \mathbf{x})\ B_i(\mathbf{a}_{-i}|\mathbf{x})\ d\mathbf{a}_{-i} - \varepsilon_i < 0$$

- For both continuous and discrete choice, best response implies the following **restrictions on the cumulative choice probability function**. For any $a^0 \in \mathcal{A}$:

$$P_i\left(a^0|\mathbf{x}\right) = F_i\left[\int \Delta\pi_i(a^0, \mathbf{a}_{-i}, \mathbf{x})\ B_i(\mathbf{a}_{-i}|\mathbf{x})\ d\mathbf{a}_{-i}\right]$$

General model of firms' beliefs

$$P_i\left(a^0|\mathbf{x}\right) = F_i\left[\int \Delta\pi_i(a^0, \mathbf{a}_{-i}, \mathbf{x})\ B_i(\mathbf{a}_{-i}|\mathbf{x})\ d\mathbf{a}_{-i}\right]$$

- These best response conditions contain all the restrictions of the model on beliefs function B_i and profit function $\Delta\pi_i$.

- Many models of competition in IO – under different types of equilibrium concepts are particular versions of this model.

- **Auctions, Bertrand competition, Cournot competition, Entry models** under different types of restrictions on firms' beliefs:
  - o **Bayesian Nash equilibrium**.
  - o **Level-K and Cognitive Hierarchy Beliefs**.
  - o **Rationalizability.**

Bayesian Nash Equilibrium

- Under Bayesian Nash Equilibrium (with independent private values):

$$B_i(\mathbf{a}_{-i} \mid \mathbf{x}) = \Pr\left(\mathbf{a}_{-i} \mid \mathbf{x}\right)$$

- This is the most commonly used solution concept in games of incomplete information in IO.
- It has received particular attention in **auction games** and in discrete choice models of **market entry**, but it has been also applied to games of quantity or price competition.

Cognitive Hierarchy and Level-K models

- Equilibrium concepts where firms have biased beliefs, that is, $B_i(\mathbf{a}_{-i} \mid \mathbf{x}) \neq \Pr(\mathbf{a}_{-i} \mid \mathbf{x})$.
- There is a finite number K of belief types that correspond to different levels of strategic sophistication.
- Believes for Level-0 can be arbitrary, $B^{(0)}$.
- Level-1 firms believe that all the other firms are level 0:

$$B^{(1)}(\mathbf{a}_{-i} \mid \mathbf{x}) = \prod_{j \neq i} F\left[\int \Delta\pi(a_j, \mathbf{a}_{-j}, \mathbf{x}) \; B^{(0)}(\mathbf{a}_{-j}|\mathbf{x}) \; d\mathbf{a}_{-j}\right]$$

Cognitive Hierarchy and Level-K models [2]

- In **Level-k model**, a level-k firm believes that all the other firms are $k-1$.

$$B^{(k)}\left(\mathbf{a}_{-i} \mid \mathbf{x}\right) \ = \ \prod_{j \neq i} F\left[\int \Delta\pi(a_j, \mathbf{a}_{-j}, \mathbf{x}) \ B^{(k-1)}(\mathbf{a}_{-j}|\mathbf{x}) \ d\mathbf{a}_{-j}\right]$$

- In **Cognitive Hierarchy model**, a level-k firm believes that all other firms come from a probability distribution over levels 0 to $k-1$.

- These models impose restrictions on beliefs.

o There is a finite number K of belief types (typically 2 or 3).
o These belief functions satisfy a hierarchical equilibrium.

Rationalizability

- The concept of *Rationalizability* (Bernheim, 1984; Pearce, 1984) imposes two restrictions on firms' beliefs and behavior.
  [A.1] Every firm is rational in the sense that it maximizes its own expected profit given beliefs.
  [A.2] This rationality is common knowledge, i.e., every firms knows that all the firms know that it knows ... that all the firms are rational.

- Aradillas-Lopez & Tamer (2008) study identification under Rationalizability.

- In a game **with multiple equilibria**, the solution concept of **Rationalizability allows for biased beliefs**.

- Each firm has beliefs that are consistent with a BNE, but these beliefs may not correspond to the same BNE.

4. IDENTIFICATION

Data

- The researcher has a sample of M local markets, indexed by m, where she observes firms' actions and state variables (**firms' choice data**):

$$\{a_{imt},\ \mathbf{x}_{mt} : i = 1, 2, ..., N;\ t = 1, 2, ..., T\}$$

- In addition to these data, the researcher may have data on some components of the profit function.

- We can distinguish three cases, from the best to the worst case scenario:

  (a) Only Choice Data
  (b) Choice data + Revenue function
  (c) Choice data + Revenue function + Cost function

Profit = Revenue - Cost

- To incorporate explicitly the researcher's information about the revenue and/or cost function, we take into account that:

$$\Delta\pi_i(a_i, \mathbf{a}_{-i}, \mathbf{x}) = \Delta r_i(a_i, \mathbf{a}_{-i}, \mathbf{x}) - \Delta c_i(a_i, \mathbf{a}_{-i}, \mathbf{x})$$

  where $\Delta r_i \equiv$ Marginal Revenue, and $\Delta c_i \equiv$ Marginal Cost.

- In most models of competition (Bertrand, Cournot, Entry), strategic interactions occur through demand and revenue but not through costs. We may incorporate this restriction:

$$\Delta c_i(a_i, \mathbf{a}_{-i}, \mathbf{x}) = \Delta c_i(a_i, \mathbf{x})$$

- We have:

  (a) Only Choice Data: Δr_i & Δc_i unknown

  (b) Choice + Revenue: Δr_i known

  (c) Choice + Revenue + Cost: Δr_i & Δc_i known

Identification Problem

$$P_i\left(a_i|\mathbf{x}\right) = F_i\left[\int \Delta r_i(a_i, \mathbf{a}_{-i}, \mathbf{x})\ B_i(\mathbf{a}_{-i}|\mathbf{x})\ d\mathbf{a}_{-i} - \Delta c_i(a_i, \mathbf{x})\right]$$

- $P_i\left(a_i|\mathbf{x}\right)$ is nonparametrically identified from the data at any value $(a_i, \mathbf{x})$ in the support of these variables.

- The researcher is interested in the identification of:
  - firms' belief functions $\{B_i(\mathbf{a}_{-i}|\mathbf{x})\}$
  - MC functions $\{\Delta c_i(a_i, \mathbf{x})\}$, and MR functions $\Delta r_i(a_i,\mathbf{a}_{-i}, \mathbf{x})$
  - Distributions of the private information, F_i

- We are interested in identification results that do not rely on parametric assumptions, especially on beliefs, because beliefs are endogenous objects.

Binary choice – Two-player game

- We start presenting identification results for the simpler version of this model: **Binary Choice, Two Players, and Known Distribution** F_i.
- We show later that these identification results extend to models with multinomial or continuous choice, N players, and nonparametric specification of F_i.
- We cam think in game of market entry, or in a **game of price competition where firms choose between a high price (**$a_i = 1$**) and a low price (**$a_i = 0$**)**.

Binary choice – Two-player game [2]

- The best response equation:

$$P_i(\mathbf{x}) = F_i\left([\Delta r_i(0,\mathbf{x}) + B_i(\mathbf{x})\ \ [\Delta r_i(1,\mathbf{x}) - \Delta r_i(0,\mathbf{x})] - \Delta c_i(\mathbf{x})]\right)$$

  $P_i(\mathbf{x}) =$ probability for the choice of high price by firm i.
  $B_i(\mathbf{x}) =$ belief probability that competitor chooses high price.
  $\Delta r_i(a_{-i},\mathbf{x}) \equiv r_i(1,a_{-i},\mathbf{x}) - r_i(0,a_{-i},\mathbf{x})$ is the marginal revenue.
  $\Delta c_i(\mathbf{x}) \equiv c_i(1,\mathbf{x}) - c_i(0,\mathbf{x})$ is the marginal cost.

- Define the quantile function $Q_i(\mathbf{x}) \equiv F_i^{-1}\left(P_i(\mathbf{x})\right)$. The best response can be described as:

$$Q_i(\mathbf{x}) = \Delta r_i(0,\mathbf{x}) + B_i(\mathbf{x})\ \ [\Delta r_i(1,\mathbf{x}) - \Delta r_i(0,\mathbf{x})] - \Delta c_i(\mathbf{x})$$

Identification with revenue and cost data

$$Q_i(\mathbf{x}) = \Delta r_i(0, \mathbf{x}) + B_i(\mathbf{x}) \; [\Delta r_i(1, \mathbf{x}) - \Delta r_i(0, \mathbf{x})] - \Delta c_i(\mathbf{x})$$

- where $Q_i(\mathbf{x})$, $\Delta r_i(0, \mathbf{x})$, $\Delta r_i(1, \mathbf{x})$, and $\Delta c_i(\mathbf{x})$ are known.
- Given this information, **under the condition that** $\Delta r_i(1, x) - \Delta r_i(0, x) \neq 0$ – i.e., the model is a game, there are strategic effects – the observed behavior of firm i reveals her beliefs about the behavior of the competitor:

$$B_i(\mathbf{x}) = \frac{Q_i(\mathbf{x}) + \Delta c_i(\mathbf{x}) - \Delta r_i(0, \mathbf{x})}{\Delta r_i(1, \mathbf{x}) - \Delta r_i(0, \mathbf{x})}$$

- This belief function can be compared to the actual choice probability of the competitor to test unbiased / rational beliefs:

$$B_i(\mathbf{x}) - P_{-i}(\mathbf{x}) = 0 \quad ?$$

- We can also test other restrictions such as Rationalizability, level-K or Cognitive Hierarchy models.

Identification with revenue but not cost data

- MR functions $\Delta r_i(0, \mathbf{x})$ and $\Delta r_i(1, \mathbf{x})$ are known to the researcher but the MC cost $\Delta c_i(\mathbf{x})$ is not known.

- Without further restrictions, the system of equations

$$Q_i(\mathbf{x}) = \Delta r_i(0, \mathbf{x}) + B_i(\mathbf{x}) \ \left[\Delta r_i(1, \mathbf{x}) - \Delta r_i(0, \mathbf{x})\right] - \Delta c_i(\mathbf{x})$$

  cannot identify the unknown functions $B_i(\mathbf{x})$ and $c_i(\mathbf{x})$.

- Without further restrictions, any Belief function (including the BNE belief) is consistent with observed behavior, $Q_i(\mathbf{x})$, given the appropriate MC function.

Identification: Firm-specific cost shifter

- **Exclusion Restriction (Firm specific cost shifter)**:
  The vector $\mathbf{x}$ has a firm-specific components that affects MC or MR of a firm but not MC or MR of other firms.

- That is, $\mathbf{x} = (\mathbf{w}, \mathbf{z}_i, \mathbf{z}_{-i})$ such that Δr_i or/and Δc_i depend on $\mathbf{z}_i$ but not on $\mathbf{z}_{-i}$:

$$\Delta c_i(\mathbf{x}) = \Delta c_i(\mathbf{w}, \mathbf{z}_i)$$

- **Examples:**

  - Firm specific input prices: wages, prices of intermediate inputs.

  - Firm specific predetermined variables affecting a firm's profit: in pricing game with menu costs, the firm's price at previous period affects the own firm's profit (its own menu cost) but not the competitor's profit.

Identification: Firm-specific cost shifter [2]

- Let $\mathbf{z}_{-i}^1$ and $\mathbf{z}_{-i}^2$ be two values for $\mathbf{z}_{-i}$. (I omit here the common variables $\mathbf{w}$):

$$\begin{cases} Q_i(\mathbf{z}_i, \mathbf{z}_{-i}^1) = r_i(0, \mathbf{z}_i) + B_i(\mathbf{z}_i, \mathbf{z}_{-i}^1) \ [r_i(1, \mathbf{z}_i) - r_i(0, \mathbf{z}_i)] - c_i(\mathbf{z}_i) \\ \\ Q_i(\mathbf{z}_i, \mathbf{z}_{-i}^2) = r_i(0, \mathbf{z}_i) + B_i(\mathbf{z}_i, \mathbf{z}_{-i}^2) \ [r_i(1, \mathbf{z}_i) - r_i(0, \mathbf{z}_i)] - c_i(\mathbf{z}_i) \end{cases}$$

- Taking the difference between these best-response equations:

$$B_i(\mathbf{z}_i, \mathbf{z}_{-i}^2) - B_i(\mathbf{z}_i, \mathbf{z}_{-i}^1) = \frac{Q_i(\mathbf{z}_i, \mathbf{z}_{-i}^2) - Q_i(\mathbf{z}_i, \mathbf{z}_{-i}^1)}{r_i(1, \mathbf{z}_i) - r_i(0, \mathbf{z}_i)}$$

- Again, as long as $r_i(1, \mathbf{z}_i) - r_i(0, \mathbf{z}_i) \neq 0$, firm i's observed behavior – her observed change in behavior when $\mathbf{z}_{-i}$ changes, $Q_i(\mathbf{z}_i, \mathbf{z}_{-i}^2) - Q_i(\mathbf{z}_i, \mathbf{z}_{-i}^1)$ – reveals her beliefs.

- We can test unbiased beliefs, or other restrictions on beliefs:

$$B_i(\mathbf{z}_i, \mathbf{z}_{-i}^2) - B_i(\mathbf{z}_i, \mathbf{z}_{-i}^1) = P_{-i}(\mathbf{z}_i, \mathbf{z}_{-i}^2) - P_{-i}(\mathbf{z}_i, \mathbf{z}_{-i}^1) \ ?$$

Identification using only firms' choice data

- This exclusion restriction can be applied to the identification of beliefs also when the researcher does not know the revenue function.

- Let $\mathbf{z}_{-i}^{1}$, $\mathbf{z}_{-i}^{2}$, and $\mathbf{z}_{-i}^{3}$ be three values for $\mathbf{z}_{-i}$.

$$\left\{ \begin{array}{l} Q_i(\mathbf{z}_i, \mathbf{z}_{-i}^{2}) - Q_i(\mathbf{z}_i, \mathbf{z}_{-i}^{1}) = \left[B_i(\mathbf{z}_i, \mathbf{z}_{-i}^{2}) - B_i(\mathbf{z}_i, \mathbf{z}_{-i}^{1}) \right] \;\; \left[r_i(1, \mathbf{z}_i) - r_i(\right. \\ \\ Q_i(\mathbf{z}_i, \mathbf{z}_{-i}^{3}) - Q_i(\mathbf{z}_i, \mathbf{z}_{-i}^{1}) = \left[B_i(\mathbf{z}_i, \mathbf{z}_{-i}^{3}) - B_i(\mathbf{z}_i, \mathbf{z}_{-i}^{1}) \right] \;\; \left[r_i(1, \mathbf{z}_i) - r_i(\right. \end{array} \right.$$

- And taking the ratio between these two differences, we have that:

$$\frac{B_i(\mathbf{z}_i, \mathbf{z}_{-i}^{2}) - B_i(\mathbf{z}_i, \mathbf{z}_{-i}^{1})}{B_i(\mathbf{z}_i, \mathbf{z}_{-i}^{3}) - B_i(\mathbf{z}_i, \mathbf{z}_{-i}^{1})} = \frac{Q_i(\mathbf{z}_i, \mathbf{z}_{-i}^{2}) - Q_i(\mathbf{z}_i, \mathbf{z}_{-i}^{1})}{Q_i(\mathbf{z}_i, \mathbf{z}_{-i}^{3}) - Q_i(\mathbf{z}_i, \mathbf{z}_{-i}^{1})}$$

Extensions

- This identification result can be extended in different ways:
  - More than two players.
  - Continuous choice games and ordered multinomial choice games
  - Nonparametric distribution of private information F_i

Multinomial or continuous choice game

- All the previous result extend to a static two-player game where the decision variable is multinomial of continuous.
- The dimension of the beliefs function is still the same as the dimension of the choice probability function.

More than two players

- Consider that there are $N > 2$ firms. The best response implies:

$$Q_i(\mathbf{x}) = \sum_{\mathbf{a}_{-i}} B_i(\mathbf{a}_{-i} \mid \mathbf{x}) \; r_i(\mathbf{a}_{-i}, \mathbf{x}) - c_i(\mathbf{x})$$

- Even if the researcher knows the MR and MC functions, there are infinite values of $B_i(\mathbf{a}_{-i} \mid \mathbf{x})$ that can rationalize the observed behavior $Q_i(\mathbf{x})$.

- However, the exclusion restriction of a firm-specific cost shifter still implies identification of beliefs with $N > 2$ players, and even when the MC and MR functions are not known to the researcher.

- All we need is that the space of the cost shifter $\mathbf{z}_{-i}$ has at least as many points of support as the points in the support of the competitors actions, $\mathbf{a}_{-i}$.

Identification with nonparametric distribution private info

- When the decision variable is continuous, there is identification of beliefs even if F_i is nonparametrically specified.

- Let $(a_i^1, \mathbf{z}_{-i}^1)$ and $(a_i^2, \mathbf{z}_{-i}^2)$ be two values of $(a_i, \mathbf{z}_{-i})$ such that $P_i(a_i^1 \mid \mathbf{z}_i, \mathbf{z}_{-i}^1) = P_i(a_i^2 \mid \mathbf{z}_i, \mathbf{z}_{-i}^2)$. This implies that:

$$Q_i(a_i^1 | \mathbf{z}_i, \mathbf{z}_{-i}^1) - Q_i(a_i^2 | \mathbf{z}_i, \mathbf{z}_{-i}^2) = 0$$

- And in turn, this implies:

$$\begin{aligned} & \int \Delta\pi_i(a_i^1, \mathbf{a}_{-i}, \mathbf{z}_i, \mathbf{z}_{-i}^1) \; B_i(\mathbf{a}_{-i} | \mathbf{z}_i, \mathbf{z}_{-i}^1) \; d\mathbf{a}_{-i} \\ - & \int \Delta\pi_i(a_i^2, \mathbf{a}_{-i}, \mathbf{z}_i, \mathbf{z}_{-i}^2) \; B_i(\mathbf{a}_{-i} | \mathbf{z}_i, \mathbf{z}_{-i}^2) \; d\mathbf{a}_{-i} \quad = 0 \end{aligned}$$

- And we can use these equations (at multiple pairs) to identify beliefs.

Full Identification of the Model

How to choose points where to impose unbiased beliefs?

- *(a) Applying the test of equilibrium beliefs.*
- *(b) Testing for the monotonicity of beliefs and using this restriction.*
- *(c) Minimization of the player's beliefs bias.*
- *(d) Most visited states.*

Identification of Cognitive Hierarchy model

- Goldfarb and Yang (2009).
- Goldfarb and Xiao (2011).

Combining choice data with firms' costs data

- Hortaçsu and Puller (2008)
- Hortaçsu et al. (2017)

Beliefs data

- DellaVigna (2009). Manski (2018).